# An amazing construction


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**Problem** Given four points, construct a square such that each side of the square, extended if necessary, passes through one of the points.

I am sure that this problem has an elegant solution using Euclidean methods. However, my solution uses methods of coordinate geometry.

Let the points be $\{P_1, P_2, P_3, P_4\}$. Choose $P_1$ to be the origin and choose the line joining it to $P_3$ to be the $x$-axis so that the coordinates of this second point are $(x_s, 0)$. Let the other two points be $P_4 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. From this there are two solutions.

Firstly, from the origin, along the $y$-axis measure a distance $x_2 - x_1$ and, perpendicular to it, a distance $x_s + y_1 - y_2$. The line joining this new point to the origin is one side of the square, the line through $(x_s, 0)$ parallel to it is another, and the two lines through $(x_1, y_1)$ and $(x_2, y_2)$, perpendicular to them complete the square.

Secondly, from the origin, along the $y$-axis measure a distance $x_1 - x_2$ and, perpendicular to it, a distance $x_s - y_1 + y_2$. The line joining this new point to the origin is one side of the square, the line through $(x_s, 0)$ parallel to it is another. As before, the two lines through $(x_1, y_1)$ and $(x_2, y_2)$, perpendicular to them complete the square.

If we choose a different pair of points to define the $x$-axis we arrive at another pair of solutions. It turns out that for a general configuration, there are six distinct squares that can be drawn through the four points. The front cover of this magazine illustrates the 48 solutions obtained by performing the two constructions on the 24 permutations of a given set of four points. Readers can verify that each square appears eight times.

**Proof** A point can be defined by its distances from two perpendicular axes $(x, y)$ and the equation of a line is often given as $y = mx + c$, where $m$ is called the gradient. Thus I can define a line by two numbers $[m, c]$. If a line has gradient $m$ then a line perpendicular to it has gradient $-1/m$.

These two definitions are linked. A line through two points $(x, y)$ and $(u, v)$ becomes

$$\left[\frac{y-v}{x-u},\ y - x\frac{y-v}{x-u}\right].$$

The intersection of two lines $[m_1, c_1]$ and $[m_2, c_2]$ becomes

$$\left(\frac{c_2-c_1}{m_1-m_2},\ m_1\frac{c_2-c_1}{m_1-m_2} + c_1\right).$$

A line through $(x, y)$ with gradient $m$ becomes $[m,\ y-mx]$, and the distance between $(x, y)$ and $(u, v)$, squared, is $(x-u)^2+(y-v)^2$.

Let

$S_1 = [m,\ 0]$ be the line through the origin $(P_1)$ with gradient $m$,

$S_3 = [m,\ -mx_s]$ be the line through $P_3$ with gradient $m$,

$S_2 = [-1/m,\ x_2/m + y_2]$ be the line through $P_2$ with gradient $-1/m$,

$S_4 = [-1/m,\ x_1/m + y_1]$ be the line through $P_4$ with gradient $-1/m$.

Denoting the intersection of $S_i$ and $S_j$ by $S_{i,j}$, let

$$\begin{aligned}
V_1 &= S_{1,2} = \left(\frac{x_2+my_2}{1+m^2},\ \frac{m(x_2+my_2)}{1+m^2}\right),\\
V_2 &= S_{2,3} = \left(\frac{x_2+m(mx_s+y_2)}{1+m^2},\ \frac{m(x_2-x_s+my_2)}{1+m^2}\right),\\
V_3 &= S_{3,4} = \left(\frac{x_1+m(mx_s+y_1)}{1+m^2},\ \frac{m(x_1-x_s+my_1)}{1+m^2}\right),\\
V_4 &= S_{1,4} = \left(\frac{x_1+my_1}{1+m^2},\ \frac{m(x_1+my_1)}{1+m^2}\right).
\end{aligned}$$

Then these four points form the vertices of a rectangle. We choose $m$ such that the sides of the rectangle are equal. It is sufficient that the squares of the sides are equal; so we equate the distances squared between $V_1$ and $V_2$ and between $V_1$ and $V_4$,

$$\begin{aligned}
&\big(x_2+m(mx_s+y_2)-(x_2+my_2)\big)^2+\big(m(x_2-x_s+my_2)-m(x_2+my_2)\big)^2\\
&\quad= \big(x_1+my_1-(x_2+my_2)\big)^2+\big(m(x_1+my_1)-m(x_2+my_2)\big)^2,
\end{aligned}$$

to obtain these two solutions,

$$m = \frac{-x_1+x_2}{x_s+y_1-y_2} \quad\text{and}\quad m = \frac{x_1-x_2}{x_s-y_1+y_2},$$

from which we can derive the two alternative constructed points:

$$P_{5,1} = (x_s+y_1-y_2,\ x_2-x_1) \quad\text{and}\quad P_{5,2} = (x_s-y_1+y_2,\ x_1-x_2).$$

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As an example, consider the set of four points defined by

$$x_1 = 2,\quad y_1 = 2,\quad x_2 = 1.3,\quad y_2 = 1.7,\quad x_s = 1.5.$$

These give

$$P_1 = (0,\ 0),\quad P_2 = (1.3,\ 1.7),\quad P_3 = (1.5,\ 0),\quad P_4 = (2,\ 2),$$

from which we derive

$$m = -0.388889, \quad P_{5,1} = (1.8,\ -0.7),$$

$$V_1 = (0.55496,\ -0.215818), \quad V_2 = (0.752011,\ 0.290885),$$

$$V_3 = (1.25871,\ 0.0938338), \quad V_4 = (1.06166,\ -0.412869),$$

and

$$m = 0.583333, \quad P_{5,2} = (1.2,\ 0.7),$$

$$V_1 = (1.70984,\ 0.997409), \quad V_2 = (2.09067,\ 0.34456),$$

$$V_3 = (2.74352,\ 0.725389), \quad V_4 = (2.36269,\ 1.37824).$$